The square of the Weyl tensor can be negative

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Abstract

We show that the square of the Weyl tensor can be negative by giving an example:

$$ds^2 = -dt^2 + 2yzdtdx + dx^2 + dy^2 + dz^2 \,.$$

This metric has the property that in a neighbourhood of the origin,

$$C^{ijkl}\, C_{ijkl} < 0 \,.$$

KEY: Sign of curvature invariants; Weyl tensor

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Let us consider the metric

$$ds^2 = -dt^2 \, + \, 2\,y\,z\,dt\,dx + dx^2 + dy^2 + dz^2 \,. \tag{1}$$

At the coordinate origin, the metric equals just the usual Minkowski metric

$$\eta_{ij} = \mathrm{diag}(-1,\, 1,\, 1,\, 1)$$

and all Christoffel symbols vanish. The only components of $g_{ij,kl}$ which do not vanish are

$$g_{01,23} = g_{10,23} = g_{01,32} = g_{10,32} = 1\,. \tag{2}$$

Therefore, at $t = x = y = z = 0$, only those components of the Riemann tensor, where all 4 indices are different from each other, can have a non-vanishing value. Example:

$$R_{0312} = -1/2\,, \qquad R^{0312} = 1/2\,, \qquad R_{0312} \cdot R^{0312} = -1/4\,. \tag{3}$$

One gets

$$R^{ijkl} R_{ijkl} < 0\,.$$

Further, the Ricci tensor vanishes there, and the Weyl tensor is equal to the Riemann tensor. Consequently, the square of the Weyl tensor is negative, and, by continuity, metric (1) represents an example of a spacetime, where

$$C^{ijkl}\, C_{ijkl} < 0\,,$$

at least in an open neighbourhood of the origin.

Here this inequality can be applied: Equation (27) of [1] reads

$$B = \left(W^s{}_{ikl}\, W^r{}_{abc}\, g_{sr}\, g^{ia}\, g^{kb}\, g^{lc} \right)^{1/2}$$

where $W^s{}_{ikl}$ denotes the Weyl tensor, which we preferred to denote by $C^s{}_{ikl}$ in the above text. In the note added to [1], the authors argued that eq. (27) could not have the character of a square root from a negative real.

The purpose of this comment is to clarify, that from the purely differential geometric point of view, this is not the case, and therefore, one needs further physical motivations for the exclusion of negative values of the square of the Weyl tensor.

Note added: Possible negative values of “quadratic” curvature invariants have also been discussed recently in [2]; there the authors proposed to call the sets where this happens “regions of gravitomagnetic dominance”.

References

[1] H.-H. v. Borzeszkowski, H.-J. Treder: On matter and metric in affine theory of gravity, Gen. Rel. Grav. **34** (2002) 1909.

[2] C. Cherubini, D. Bini, S. Capozziello, R. Ruffini: Second order scalar invariants of the Riemann tensor: applications to black hole spacetimes, Int. J. Mod. Phys. **D 11** (2002) 827.